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THE EXPONENTIAL GROWTH OF CODIMENSIONS FOR CAPELLI IDENTITIES

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ABSTRACT

By applying the theorem that every positive integer is a sum of four squares, we calculate the exponential growth of the codimensions for the relatively free algebra satisfying Capelli identities.

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Introduction

Let F be a field of characteristic zero, $F\langle x\rangle = F\langle x_1, x_2, \ldots\rangle$ the free associative algebra of non-commutative polynomials. Let V be a variety of P.I. algebras, given by a T ideal of identities $I \subseteq F\langle x\rangle$. Let

$$P_n(V) = \frac{V_n}{V_n \cap I}$$

denote the multilinear polynomials of degree n in $x_1, \ldots, x_n$ in the relatively free algebra in V; then

$$c_n(V) = \dim\left(\frac{V_n}{V_n \cap I}\right)$$

is the nth codimension of V.

It was proved in [GZ2] that $\lim_{n\to\infty} \sqrt[n]{c_n(V)}$ always exist and is a non-negative integer (see also [GZ1]). It is denoted by $\mathrm{Exp}(V) = \lim_{n\to\infty} \sqrt[n]{c_n(V)}$.

The polynomial

$$\begin{aligned} c_{m+1} &= c_{m+1}[x_1, \ldots, x_{m+1}; y_1, \ldots, y_m] \\ &= \sum (-1)^\sigma x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_m x_{\sigma(m+1)} \end{aligned}$$

is called the $(m+1)$th Capelli polynomial. Let C_{m+1} denote the set of the 2^m polynomials obtained from c_{m+1} by deleting a subset of the y's (i.e. substituting $y_i \to 1$ for such a subset). Let I be the T ideal generated by C_{m+1}, with the corresponding variety U_{m+1}. Capelli polynomials play a major role in P.I. theory and our objective here is to calculate $\mathrm{Exp}(U_{m+1})$. These polynomials were first introduced by Razmyslov [Ra] in his construction of central polynomials for $k \times k$ matrices. It is easy to show that for a finite dimensional algebra A, if $\dim A = m$ then $A \in U_{m+1}$ (i.e. A satisfies $f \equiv 0$ for any $f \in C_{m+1}$). Moreover, any finitely generated P.I. algebra A satisfies C_{m+1} for some m. See, for example, Theorem 2.2 in [K].

Let S_n be the symmetric group. Then $P_n(V)$ is a left FS_n module, hence defines the S_n character

$$\chi_n(V) = \chi_{S_n}(P_n(V)),$$

known as the cocharacter of V [R1]. It is well known that

$$\chi_n(V) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$$

where $\lambda \vdash n$ (i.e. λ is a partition of n), χ_λ the corresponding irreducible S_n character and m_λ the corresponding multiplicity of χ_λ in $\chi_n(V)$. The nth cocharacter $\chi_n(A)$ of a P.I. algebra is defined similarly.

In terms of cocharacters, Capelli identity is almost the general case in P.I. theory. Recall the notation $H(k,\ell;n) = \{(\lambda_1, \lambda_2, \ldots) \vdash n |\ \lambda_{k+1} \leq \ell\}$, $H(k,\ell) = \bigcup_n H(k,\ell;n)$. For a general (P.I.) variety V there exist k, ℓ such that $\{\chi_n(V)\} \subseteq H(k,\ell)$, i.e. $\chi_n(V) = \sum_{\lambda \in H(k,\ell;n)} m_\lambda \chi_\lambda$ [AR]. However, A satisfies C_{m+1} if and only if $\{\chi_n(A)\} \subseteq H(m,0)$ [R2].

Our objective here is to calculate $\mathrm{Exp}(U_{m+1})$. Since m_λ's are always polynomially bounded [B], [BR2], it follows from [R3] that $\mathrm{Exp}(U_{m+1}) \leq m$. In case $m = k^2$, it follows from [R4] that

$$\mathrm{Exp}(U_{k^2+1}) = k^2.$$

For an arbitrary m we prove here the following

THEOREM:
(1) $m - 3 \leq \mathrm{Exp}(U_{m+1}) \leq m$.
(2) $\mathrm{Exp}(U_{m+1}) = \max\{a_1, a_2, a_3, a_4\}$ *where*

$$a_j = \max\{d_1^2 + \cdots + d_j^2 |\ d_1, \ldots, d_j \text{ are positive integers and } d_1^2 + \cdots + d_j^2 + j \leq m+1\}. \tag{1}$$

(3) $\mathrm{Exp}(U_{m+1}) = m$ *if and only if* $m = q^2$ *for some* q.

The proof, which is given below, applies, in an essential way, the classical theorem that every $0 \leq m \leq \mathbb{Z}$ is a sum of at most four non-zero squares.

Note that the codimension growth of U_{m+1} does not depend on extensions of the ground field. Therefore, we assume below that F is an algebraically closed field of characteristic zero.

LEMMA 1: *Let $B_1, \ldots, B_k$ be finite-dimensional simple algebras over F such that $B = B_1 \oplus \cdots \oplus B_k$ is a subalgebra of finite-dimensional algebra A with Jacobson radical $J = J(A)$. Also let $d_i = \dim B_i$, $i = 1, \ldots, k$, and $d = d_1 + \cdots + d_k$. Suppose that*

$$B_1 J B_2 J \cdots J B_k \neq 0 \tag{2}$$

in A. Then $A \notin U_{d+k-1}$.

Proof: Since F is algebraically closed, any B_i is isomorphic to some matrix algebra over F. The inequality (2) implies that there exist $c_1, \ldots, c_{k-1} \in J$, $e_i \in B_i$, $i = 1, \ldots, k$ such that

$$e_1 c_1 e_2 c_2 \cdots c_{k-1} e_k \neq 0, \tag{3}$$

each e_i is some matrix unit from B_i, and also

$$1_i c_i = c_i 1_{i+1} = c_i \tag{4}$$

where $1_i \in B_i$ is an identity element from B_i.

Now consider one of these algebras B_i and fix its basis $u_1^i, \ldots, u_{d_i}^i$ from matrix units. Obviously, one can choose matrix units $a_1^i, \ldots, a_{d_i+1}^i \in B_i$ such that

$$f_i(u_1^i, \ldots, u_{d_i}^i, a_1^i, \ldots, a_{d_i+1}^i) = a_1^i u_1^i a_2^i \cdots a_{d_i}^i u_{d_i}^i a_{d_i+1}^i = b^i \neq 0, \tag{5}$$

but

$$f_i(u_{\sigma(1)}^i, \ldots, u_{\sigma(d_i)}^i, a_1^i, \ldots, a_{d_i+1}^i) = 0 \tag{6}$$

for any non-trivial permutation $\sigma \in S_{d_i}$. Since b^i is a matrix unit, there exist b_1^i, b_2^i such that $b_1^i b^i b_2^i = e_i$.

By our choice, using (4), we have

$$b_2^i t b_1^{i+1} = 0 \quad \text{if } t \in B_1 \oplus \cdots \oplus B_k \quad \text{or} \quad t = c_j \quad \text{for } j \neq i. \tag{7}$$

(Clear if $t \in B_1 \oplus B_2 \cdots \oplus B_k$. If $t = c_j$, then $t = 1_j c_j 1_{j+1}$ and $1_j B_i = B_i 1_j = 0$ if $j \neq i$.) Also

$$b_1^i t b_2^i = 0 \quad \text{if } t = c_j \quad \text{or} \quad t \in B_s, \quad s \neq i \tag{8}$$

(similar reasons!).

We now construct a polynomial which is alternating on $d+k-1$ variables and has a non-zero value on A. Define it as $\mathrm{Alt}(g)$ for

$$\begin{aligned} g = & g(x_1^1, \ldots, x_{d_1}^1, \ldots, x_1^k, \ldots, x_{d_k}^k, y_1^1, \ldots, y_{d_1+1}^1, \ldots, y_1^k, \ldots, y_{d_k+1}^k, \\ & z_1, \ldots, z_{k-1}, w_1, \ldots, w_{2d}) \\ = & w_1 f_1 w_2 z_1 w_3 f_2 w_4 z_2 \cdots z_{k-1} w_{2k-1} f_k w_{2k}, \end{aligned}$$

where $f_i = f_i(x_1^i, \ldots, x_{d_i}^i, y_1^i, \ldots, y_{d_i+1}^i)$ is the monomial constructed earlier for B_i and Alt means an alternation on all $x_1^1, \ldots, x_{d_k}^k, z_1, \ldots, z_{k-1}$.

It follows from (4) that $1_i c_j = c_j 1_s = 0$ if $i \neq j$, $s \neq j+1$. Hence, from (5), (6), (7), (8) and (3) it follows that for the substitution φ such that

$$\begin{aligned} & \varphi(x_j^i) = u_j^i, \quad \varphi(y_j^i) = a_j^i, \quad \varphi(z_i) = c_i, \quad \varphi(w_{2i-1}) = b_1^i, \\ & \varphi(w_{2i}) = b_2^i \end{aligned}$$

we get

$$\varphi(\mathrm{Alt}(g)) = \varphi(g) = e_1 c_1 e_2 c_2 \cdots c_{k-1} e_k \neq 0.$$

This last inequality proves Lemma 1. ∎

LEMMA 2: *For any $k \geq 1$ and for any integers $q_1, \ldots, q_k \geq 1$ there exists a finite-dimensional algebra $A = B + J$ with a semisimple subalgebra B and the Jacobson radical J such that*

(1) $B = B_1 \oplus \cdots \oplus B_k$.
(2) *All B_i's are simple with* $\dim B_i = q_i^2$.
(3) $\mathrm{Exp}(A) = q_1^2 + \cdots + q_k^2 = \dim B$.
(4) $A \in U_{d+k}$ *where* $d = \mathrm{Exp}(A)$.

Proof: We construct A as block upper-triangular matrices

$$\begin{pmatrix} B_1 & & & * \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix}$$

where B_i are the $q_i \times q_i$ matrices over the field F. Then $B = B_1 \oplus \cdots \oplus B_k$ is a semisimple subalgebra in A and J consists of all block strictly upper-triangular matrices.

By [GZ1] the exponent of A is exactly $\dim B = q_1^2 + \cdots + q_k^2$ since

$$B_1 J B_2 J \cdots J B_k \neq 0.$$

On the other hand, any multilinear polynomial $f = f(x_1, \ldots, x_{d+k}, y_1, y_2, \ldots)$ alternating on the $d+k$ variables $x_1, \ldots, x_{d+k}$ is equal to zero identically on A. Indeed, by multilinearity we can consider only substitutions φ: $x_i \to \overline{x}_i$, $y_j \to \overline{y}_j$, $\overline{x}_i, \overline{y}_j \in A$, such that $\overline{x}_i \in B$ or $\overline{x}_i \in J$ for $1 \leq i \leq d+k$. However, if $\overline{x}_i \in B$ for $d+1$ or more $\overline{x}$'s then $\varphi f = f(\overline{x}, \overline{y}) = 0$ (by alternation). On the other hand, if we substitute more than $k-1$ elements from J then $\varphi f = 0$ since $J^k = 0$. Hence A satisfies all Capelli identities of rank $d+k$. ∎

From [R2], [BR1], [R3] and [GZ1] follows

LEMMA 3: *The exponent t of the variety U_{m+1} exists, is an integer and does not exceed m.*

The proof of the Theorem: By Lemma 3 the exponent t of U_{m+1} exists and $t = \mathrm{Exp}(U_{m+1}) \leq m$.

If we define integers a_j, $j \geq 1$ as in (1), then by Lemma 2 one has $t \geq a_j$ and therefore

$$t \geq a_0 = \max\{a_1, a_2, a_3, a_4\}. \tag{9}$$

We now apply the classical theorem (proved by Lagrange more than 200 years ago) stating that any positive integer is the sum of at most four squares of integers. (For the history of that theorem, see for example [NZM].)

Consider such a decomposition for $m-3$:

$$m-3=q_1^2+\cdots+q_k^2, \qquad q_1,\ldots,q_k\geq 1, \quad 1\leq k\leq 4. \tag{10}$$

It follows from (10) that

$$q_1^2+\cdots+q_k^2+k\leq q_1^2+\cdots+q_k^2+4\leq m+1.$$

Applying Lemma 2 again we see that one of $a_1,\ldots,a_4$ is not less than $m-3$. Hence $a_0\geq m-3$ and $t\geq m-3$ by (9), and the first statement of the theorem is proved.

To prove the second statement recall that any variety with a Capelli identity can be generated by some finite-dimensional algebra, i.e. $U_{m+1}=\text{var}(A)$, $\dim A<\infty$ (see Theorem 2.2 in [K]). In particular $\text{Exp}(A)$ should be equal to t. It was shown in [GZ1] that in this case there exists a family of simple subalgebras $B_1,\ldots,B_k$ such that $B_1\oplus\cdots\oplus B_k$ is a semi-simple subalgebra in A and

$$B_1JB_2J\cdots JB_k\neq 0,$$

where J is the Jacobson radical of A. In addition, $\text{Exp}(A)=t=\dim B_1+\cdots+\dim B_k$.

Now $A\in U_{m+1}$ and, by Lemma 1, $A\notin U_{t+k-1}$. Hence $t+k-1$ is strictly less than $m+1$, i.e. $t+k-1\leq m$. It follows that $t\leq m+1-k$.

Now if $k\geq 5$ then

$$t\leq m+1-k\leq m+1-5=m-4<m-3,$$

which contradicts the first statement of the theorem. If $k\leq 4$ then denote $\dim B_i=q_i^2$. Recall that our field is algebraically closed, so any simple finite dimensional algebra is a matrix algebra. By definition of $a_1,\ldots,a_4$ one has

$$t=q_1^2+\cdots+q_k^2\leq a_k\leq a_0. \tag{11}$$

Comparing (9) and (11) we get the second statement of the theorem.

Finally, for $m=q^2$ it follows from (1) that $a_1=q^2=m$ and $\text{Exp}(U_{m+1})=m$. If $m\neq q^2$, then $a_1+1<m+1$ and $a_1\leq m-1$. Also a_2,a_3,a_4 cannot be greater

than $m-1$ by definition. This proves the inequality $\operatorname{Exp}(U_{m+1}) \leq m-1 \neq m$ in the case $m \neq q^2$. ∎

We conclude by showing, by examples, that the estimation

$$m-3 \leq \operatorname{Exp}(U_{m+1}) \leq m$$

cannot be improved.

One can easily check that $\operatorname{Exp}(U_{m+1}) = m$ if $m = 4$, $\operatorname{Exp}(U_{m+1}) = m-1$ if $m = 3$, $\operatorname{Exp}(U_{m+1}) = m-2$ if $m = 7$. Direct computations based on the 2nd statement of the theorem show that $\operatorname{Exp}(U_{58}) = 54 = 57-3$.

Remark: As mentioned in the introduction, Capelli identities are a partial case of the "hook condition" and for arbitrary variety V there is an infinite hook $H(k,\ell)$ such that $\chi_n(V)$ lies in $H(k,\ell)$.

Since U_{m+1} corresponds exactly to $H(m,0)$, our theorem shows that the difference between the exponent t of U_{m+1} and the size m of this strip cannot be more than 3.

Define $U_{k,\ell}$ as a variety given by

$$m_\lambda = 0 \quad \text{in } \chi_n(U_{k,\ell}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

if $\lambda_{k+1} > \ell$. In other words, $U_{k,\ell}$ is the biggest variety whose cocharacter lies in the hook $H(k,\ell)$. We say that $k+\ell$ is the size of hook $H(k,\ell)$.

It follows from [BR1], [BR2] that $\operatorname{Exp}(U_{k,\ell}) \leq k+\ell$, and the following question arises: is there an upper bound for $k+\ell-\operatorname{Exp}(U_{k+\ell})$ which is independent of k and ℓ?

It is known that any variety can be generated by the Grassman envelope $G(A)$ of some finite-dimensional $\mathbb{Z}_2$-graded algebra A [K, Theorem 2.3]. On the other hand, it was proved in [GZ2] that $\operatorname{Exp}(G(A))$ exists and is an integer. Moreover, it was shown that the nth cocharacter of $G(A)$ "asymptotically coincides" with some hook $H(r,t)$, where $r = \dim B_0$, $t = \dim B_1$, $\operatorname{Exp}(G(A)) = r+t$ and $B = B_0 + B_1$ is a semisimple $\mathbb{Z}_2$-graded subalgebra in A. It follows from Kemer's classification [K] that for any $\mathbb{Z}_2$-graded semisimple algebra we have

$$\dim B_1 \leq \dim B_0.$$

Hence, $t \leq r$ and $\operatorname{Exp}(U_{k,\ell})$ cannot be more than $2k$. In particular, $\operatorname{Exp}(U_{1,\ell}) = 2$ for any $\ell \geq 1$. It shows that there is no such upper bound for the difference $(k+\ell) - \operatorname{Exp}(U_{k,\ell})$ which is independent of $k+\ell$.

Addendum

We thank S. Ahlgren for his essential help with this note.

Given $m \in \mathbb{Z}_+$, let

$$a_j(m) = \max\{d_1^2 + \cdots + d_j^2 |\, d_1, \ldots, d_j \in \mathbb{Z}_+ \text{ and } d_1^2 + \cdots + d_j^2 \le m + 1 - j\}.$$

The classical theorem that every positive integer is a sum of at most four squares clearly implies that $a_4(m) = m - 3$.

Define: $b(m) = \max\{a_1, \ldots, a_4\}$.

It was proved in this paper that $b(m)$ is the exponential rate of growth of the codimensions of the Capelli identity $c_{m+1}[x_1, \ldots, x_{m+1}; y_1, \ldots, y_n]$.

Denote $W_i = \{m \ge 1 |\, b(m) = m - i\}, \quad 0 \le i \le 3$.

The **natural densities** of the W_i's are given below.

Given $W \subseteq \mathbb{Z}_+$, the natural density $d(W)$ is defined as

$$d(W) = \lim_{n \to \infty} \frac{|\{k \in W |\, k \le n\}|}{n}.$$

THEOREM: *Each of the above sets $W_0, \ldots, W_3$ is infinite. Moreover, they have the following natural densities:*

$$d(W_0) = 0, \quad d(W_1) = 0, \quad d(W_2) = 5/6, \quad d(W_3) = 1/6.$$

Proof: We have $\mathbb{Z}_+ = W_0 \cup \cdots \cup W_3$, a disjoint union.

(1) Clearly, $W_0 = \{n^2 |\, n \ge 1\}$ is infinite, with density $d(W_0) = 0$.

(2) $W_1 = \{n |\, n \text{ is not a square and } n - 1 = a^2 + b^2\}$. Trivially, W_1 is infinite [let $m = n^2 + 1 \in W_0 + 1$, then $m \notin W_0$ and $m \in W_1$, so $W_1 \supseteq W_0 + 1$].

Let $\prod_2 = \{a^2 + b^2 |\, a, b \in \mathbb{Z}\}$, then $W_1 \subseteq \prod_2 + 1$:

$$m \in W_1 \Rightarrow m - 1 \in \prod\nolimits_2 \Rightarrow m \in \prod\nolimits_2 +1. \text{ Let } \prod\nolimits_2(x) = \{t \in \prod\nolimits_2 |\, t \le x\}.$$

By a classical result of Edmund Landau [1908],

$$\left|\prod\nolimits_2(x)\right| \ll c \cdot \frac{x}{\sqrt{\log x}}.$$

Thus $d(W_1) = 0$.

(3) Given $A \subseteq \mathbb{Z}_+$, let $A' = \mathbb{Z}_+ - A$ denote its complement. Let

$$\prod\nolimits_3 = \{a^2 + b^2 + c^2 |\, a, b, c \in \mathbb{Z}\}.$$

Then

$$W_2 = \left\{n |\, n - 2 \in \prod\nolimits_3 \text{ and } n \notin W_0 \cup W_1\right\} = \left\{n |\, n - 2 \in \prod\nolimits_3\right\} \cap W_0' \cap W_1'.$$

Since $d(W_0) = d(W_1) = 0$, hence $d(W_2) = d(\{n|\ n-2 \in \prod_3\}) = d(\prod_3 +2) = d(\prod_3)$.

By a classical theorem of Legendre [1798], m is *not* a sum of three squares (i.e. $m \in \prod'_3$) if and only if $m = 4^n(8k+7)$.

The density of $\prod'_3$ is $d(\prod'_3) = \frac{1}{8}(1 + \frac{1}{4} + (\frac{1}{4})^2 + \cdots) = \frac{1}{8} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{6}$.

Thus $d(W_3) = 1 - d(\prod'_3) = \frac{5}{6}$.

(4) Since $\mathbb{Z}_{\geq 1} = W_0 \cup \cdots \cup W_3$ is a disjoint union, clearly

$$d(W_3) = 1 - [d(W_0) + d(W_1) + d(W_2)] = 1 - \frac{5}{6} = \frac{1}{6}.$$

In particular, $d(W_3) = \frac{1}{6}$. ∎

References

[AR] S. Amitsur and A. Regev, *P.I. algebras and their cocharacters*, Journal of Algebra **78** (1982), 248–254.

[B] A. Berele, *Homogeneous polynomial identities*, Israel Journal of Mathematics **42** (1982), 258–272.

[BR1] A. Berele and A. Regev, *Applications of Hook Young diagrams to P.I. algebras*, Journal of Algebra **82** (1983), 559–567.

[BR2] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Advances in Mathematics **64** (1987), 118–175.

[GZ1] A. Giambruno and M. Zaicev, *On codimension growth of finitely generated associative algebras*, Advances in Mathematics (to appear).

[GZ2] A. Giambruno and M. Zaicev, *Exponential codimension growth of P.I. algebras: an exact estimate*, Advances in Mathematics, to appear.

[K] A. Kemer, *Ideals of Identities of Associative Algebras*, American Mathematical Society Translations **87** (1991).

[NZM] I. Niven, H. Zuckerman and H. Montgomery, *An Introduction to the Theory of Numbers*, 5th edition, Wiley, New York, 1991.

[Ra] Yu. P. Razmyslov, *On the Kaplansky problem*, Izvestiya Akademii Nauk SSSR (Ser. Mat.) **37** (1973), 483–501 (Russian).

[R1] A. Regev, *The T ideal generated by the standard identity $S_3[x_1, x_2, x_3]$*, Israel Journal of Mathematics **26** (1977), 105–125.

[R2] A. Regev, *Algebras satisfying Capelli identity*, Israel Journal of Mathematics **33** (1979), 149–154.

[R3] A. Regev, *Asymptotic values for degrees associated with strips of Young diagrams,* Advances in Mathematics **41** (1981), 115–136.

[R4] A. Regev, *Codimensions and trace codimensions of matrices are asymptotically equal,* Israel Journal of Mathematics **47** (1984), 246–250.